§ Basics of Higher Cats III

§ 1 Recap on Straightening

why we need it:
most important tool in higher Cat. Theory; gives us
- lots of functors : $R \longmapsto {}_R\mathrm{Mod}$
- homotopy coherent algebra : sections + prop

$$\begin{tikzcd} \mathrm{Assoc} \arrow[r, dashed] \arrow[dr, equal] & \mathcal{C}^{\otimes} \arrow[d] \arrow[dl, phantom, "\simeq"] \\ & \mathrm{Assoc} \end{tikzcd}$$

§ 2 Kan extensions

"extend" given functors universally

e.g. $\mathrm{QCoh}(-) : \mathrm{PreStk} \longrightarrow \widehat{\mathrm{Cat}}$

needed heavily later on : convergent, (locally) of finite type prestacks, coconnective prestacks

§ 3 Cofinality

Simplify computations for (co) limits, Kan extensions

§1 Straightening

Idea

Given $\Phi : E \to B$ we may consider its fibers (for $b : B$)

$$\begin{tikzcd} E_b \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d] \\ \{b\} \arrow[r] & B \end{tikzcd}$$

and ask: For which $\Phi : E \to B$ is

$$B \ni b \longmapsto E_b \in \mathrm{Cat}$$

a functorial... note that $\exists$ two ways to be functorial: covariantly or contravariantly

If $\Phi : E \to B$ is

- a coCartesian fibration $\Longrightarrow$ $B \to \mathrm{Cat}$, $b \mapsto E_b$
- a Cartesian fibration $\Longrightarrow$ $B^{op} \to 1\mathrm{Cat}$, $b \mapsto E_b$

Straightening Unstraightening then says

$$(\mathrm{coCart}/B)_{strict} \underset{\sim}{\rightleftarrows} \mathrm{Fun}(B, 1\mathrm{Cat})$$

$$(\mathrm{Cart}/B)_{strict} \underset{\sim}{\rightleftarrows} \mathrm{Fun}(B^{op}, 1\mathrm{Cat})$$

Let us spell this out again:

def $\Phi : E \to B$

- A morphism $e \xrightarrow{f} \tilde{e}$ is called Φ-cocartesian

if $\forall \bar{e} \in E$:

action of Φ on mapping spaces

$$\begin{tikzcd}
\mathrm{Map}_E(\tilde{e}, \bar{e}) \arrow[r, "f^*"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}_E(e, \bar{e}) \arrow[d] \\
\mathrm{Map}_B(\Phi\tilde{e}, \Phi\bar{e}) \arrow[r, "(\Phi f)^*"'] & \mathrm{Map}_B(\Phi e, \Phi\bar{e})
\end{tikzcd}$$

is a pullback square.

- A morphism $e \xrightarrow{f} \tilde{e}$ in E is called Φ-Cartesian if

$\forall \bar{e} \in E$

$$\begin{tikzcd}
\mathrm{Map}_E(\bar{e}, e) \arrow[r, "f_*"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}_E(\bar{e}, \tilde{e}) \arrow[d] \\
\mathrm{Map}_B(\Phi\bar{e}, \Phi e) \arrow[r] & \mathrm{Map}_B(\Phi\bar{e}, \Phi\tilde{e})
\end{tikzcd}$$

is a pullback square.

Rmk: $f: e \to \tilde{e}$ being Φ-coCartesian means:

$$
\begin{tikzcd}
e \arrow[r, "f"] \arrow[dr, "\forall h"'] & \tilde{e} \arrow[d, dashed, "\exists ! g_\alpha"] \arrow[dl, phantom, "\simeq"] \\
& \bar{e}
\end{tikzcd}
$$

$$\downarrow \Phi$$

$$
\begin{tikzcd}
\Phi e \arrow[r, "\Phi f"] \arrow[dr, "\Phi h"'] & \Phi \tilde{e} \arrow[d, "\forall \alpha"] \arrow[dl, phantom, "\simeq"] \\
& \Phi \bar{e}
\end{tikzcd}
$$

$\Phi g_\alpha \simeq \alpha$

unique up to a contractible space of choices.

Rmk

Note that the definition of a Φ-Cocartesian morphism is <u>not</u> circular i.e

Can define $f^*: \mathrm{Map}_E(\tilde{e}, \bar{e}) \to \mathrm{Map}_E(e, \bar{e})$ without straightening/unstraightening ...

Idea: Use "Axiom of ∞-cats"

$$\mathrm{Fun}([2], E) \xrightarrow{\sim} \mathrm{Fun}([1], E) \times_E \mathrm{Fun}([1], E)$$

\begin{tikzcd}
 & \overset{12}{\bullet} \arrow[dr] & & & & \overset{12}{\bullet} \arrow[dr] & \\
\bullet \arrow[ur] \arrow[rr] \arrow[urr, phantom, "\simeq"] & & \bullet \arrow[rr, "\sim"'] & & \bullet \arrow[ur] & & \bullet
\end{tikzcd}

def $\Phi : E \to B$ is called

- coCartesian fib. if for every $e \in E$ and every morphism $\beta : b \to \tilde{b}$ w/ $b \simeq \Phi e$, there exists a Φ-coCartesian lift of β:

$$\mathrm{lift}_0^{\Phi}(e, \beta) : e \longrightarrow \beta_! e$$

notation for the codomain

i.e $\mathrm{lift}_0^{\Phi}(e,\beta) : e \to \beta_! e$ is Φ-CoCart. and

$$\Phi\left(\mathrm{lift}_0^{\Phi}(e,\beta)\right) \simeq \beta.$$

- Cartesian fibration if for all $\tilde{e} \in E$ and all $\beta : b \to \tilde{b}$ w/ $\tilde{b} \simeq \Phi\tilde{e}$, there is a Φ-Cartesian lift

$$\mathrm{lift}_1^{\Phi}(\tilde{e}, \beta) : \underbrace{\beta^*\tilde{e}}_{\text{notation for domain}} \longrightarrow \beta$$

Rmk Many equivalent (model independent) def of (co)Cartesian fibrations ... see e.g. Riehl–Verity Elements of ∞-Cats ...

A conceptually very nice equivalent def is : $\Phi : E \longrightarrow B$ is a CoCartesian fib

$$\text{if} \quad \vec{ev}_0^{\Phi} : E^{[1]} \longrightarrow E \vec{\times}_{\Phi} B \text{ has a}$$

left adjoint section ... see also Gromen Cisinski etc. Formalization of Higher Cats

Recall For $B : 1\mathrm{Cat}$, one defines

$$\begin{tikzcd} 1\mathrm{Cat}/B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1], 1\mathrm{Cat}) \arrow[d] \\ * \arrow[r] & 1\mathrm{Cat} \end{tikzcd}$$

Objects : functors $E \longrightarrow B$

morphisms :

$$\begin{tikzcd} E \arrow[rr, dashed, "\phi"] \arrow[dr] & \arrow[d, phantom, "\simeq"] & \tilde{E} \arrow[dl] \\ & B & \end{tikzcd}$$

def

- $\mathrm{CoCart}_{/B} \underset{\text{full}}{\subset} 1\mathrm{Cat}_{/B}$ full subcat. on coCartesian fibrations over B.

- $\mathrm{Cart}_{/B} \underset{\text{full}}{\subset} 1\mathrm{Cat}_{/B}$ analogously...

- $(\mathrm{CoCart}_{/B})_{\text{strict}} \underset{1\text{-full}}{\subset} \mathrm{CoCart}_{/B}$

1-full subcategory on those morphisms

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr, "\Phi"'] & \arrow[d, phantom, "\simeq"] & \tilde{E} \arrow[dl, "\tilde{\Phi}"] \\ & B & \end{tikzcd}$$

such that ϕ sends Φ-coCartesian morphisms to $\tilde{\Phi}$-coCartesian morphisms.

- Analogously: $(\mathrm{Cart}/B)_{\text{strict}} \underset{1\text{-full}}{\subset} 1\mathrm{Cat}/B$

- $0\text{-}\mathrm{CoCart}/B \underset{\text{full}}{\subset} \mathrm{CoCart}/B$ are those coCartesian fibrations $E \xrightarrow{\Phi} B$ such that E_b is a space for every $b \in B$.

Note:

$$\begin{array}{c} 0\text{-}\mathrm{CoCart}/B \cap (\mathrm{CoCart}/B)_{\text{strict}} \\ \Big\downarrow{\scriptstyle \int \, \wr} \\ 0\mathrm{CoCart}/B \end{array} \quad \text{is an equivalence}$$

- Analogously: $0\mathrm{Cart}/B \underset{\text{full}}{\subset} \mathrm{Cart}/B$

Thm

There are equivalences

$$
\begin{tikzcd}
\mathrm{Fun}(B, 1\mathrm{Cat}) \arrow[r, shift left, "\mathrm{Un}"] & (\mathrm{CoCart}/B)_{\mathrm{strict}} \arrow[l, shift left, "\mathrm{St}", "\sim"'] \\
\mathrm{Fun}(B, \mathrm{Spc}) \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, shift left] & 0\text{-}\mathrm{CoCart}/B \arrow[u, phantom, sloped, "\subseteq"] \arrow[l, shift left, "\sim"']
\end{tikzcd}
$$

contravariantly natural in B wrt/ precomposition on the left and pullback on the right.

Passing to opposite Cats:

$$
\begin{tikzcd}
\mathrm{Fun}(B^{op}, 1\mathrm{Cat}) \arrow[r, shift left, "\mathrm{Un}"] \arrow[r, phantom, "\simeq"] & (\mathrm{Cart}/B)_{\mathrm{strict}} \arrow[l, shift left, "\mathrm{St}"] \\
\mathrm{Fun}(B^{op}, \mathrm{Spc}) \arrow[u, phantom, sloped, "\subset"] \arrow[r, shift left] \arrow[r, phantom, "\simeq"] & 0\text{-}\mathrm{Cart}/B \arrow[u, phantom, sloped, "\subset"] \arrow[l, shift left]
\end{tikzcd}
$$

More explicitly:

Given $\Phi : E \to B$ coCartesian fibration, we obtain

$$
\begin{tikzcd}
B \arrow[r, "\mathrm{St}_\Phi"] & 1\mathrm{Cat} \\
b \arrow[d, "\beta"'] \arrow[r, mapsto] & E_b \arrow[d, "\beta_!"] \\
\tilde{b} \arrow[r, mapsto] & E_{\tilde{b}}
\end{tikzcd}
$$

action on morphism spaces is induced by Φ-coCartesian lifts.

Conversely: A functor

$$
\begin{tikzcd}
F : B \arrow[r] & 1\mathrm{Cat} \\
b \arrow[d, "\beta"'] \arrow[r, mapsto] & Fb \arrow[d, "F_\beta"] \\
\tilde{b} \arrow[r, mapsto] & F\tilde{b}
\end{tikzcd}
$$

gives rise to a CoCartesian fib.

$$Un_F : B^{\int} \longrightarrow B$$

where $B^{\int}$ is informally described by:

- objects: $(b \in B, x : Fb) \iff \begin{tikzcd} * \arrow[d, "x"] \\ Fb \end{tikzcd}$

- morphisms:

$$
\begin{tikzcd}
b \arrow[d, "\beta"'] \\
\tilde{b}
\end{tikzcd}
\qquad
\begin{tikzcd}
& * \arrow[dl, "x"'] \arrow[dr, "\tilde{x}"] & \\
Fb \arrow[rr, "F_\beta"'] \arrow[rr, phantom, bend left=20, "\Uparrow \xi"] & & F\tilde{b}
\end{tikzcd}
\iff
F_\beta(x) \xrightarrow{\xi} \tilde{x}
$$

$\uparrow$ (Un_F-CoCart $\iff$ ξ iso)

The functor

$$
\begin{tikzcd}
Un_F : B^{\int} \arrow[r] & B \\
(b, x) \arrow[r, mapsto] & b
\end{tikzcd}
$$

does the obvious thing.

Rmk There exists a universal coCartesian fib.

$$\begin{tikzcd} 1\mathrm{Cat}_\bullet \arrow[d] \\ 1\mathrm{Cat} \end{tikzcd}$$

such that

$$\begin{tikzcd} B^{\int} \arrow[r] \arrow[d, "\mathrm{Un}_F"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1\mathrm{Cat}_\bullet \arrow[d, "\pi^{univ}"] \\ B \arrow[r, "F"'] & 1\mathrm{Cat} \end{tikzcd}$$

"Any coCartesian fib. is obtained as a pullback from π^{univ}".

Examples:

- $\mathcal{C}$ cat, $c \in \mathcal{C}$
  - $\mathcal{C}_{c/} \longrightarrow \mathcal{C}$ coCartesian fib
    $\updownarrow$ St
    $\mathcal{C}(c,-) : \mathcal{C} \to \mathrm{Spc}$

    $\mathcal{C}_{/c} \longrightarrow \mathcal{C}$ cartes. fib
    $\updownarrow$ St
    $\mathcal{C}(-,c) : \mathcal{C}^{op} \to \mathrm{Spc}$

  - monoidal cats will be certain coCartesian fib
    $$\mathcal{C}^{\otimes} \to \Delta^{op}$$

  - $\mathcal{C}$ monoidal category, $\mathcal{C}^{+}$ a left $\mathcal{C}$-module category.
    One defines a cartesian fibration
    $$\mathrm{Alg}_{+}\mathrm{Mod}(\mathcal{C}, \mathcal{C}^{+}) \longrightarrow \mathrm{Alg}(\mathcal{C})$$
    which straightens to
    $$\mathrm{Alg}(\mathcal{C})^{op} \longrightarrow \widehat{\mathrm{Cat}}$$
    $$A \longmapsto {}_{A}\mathrm{Mod}(\mathcal{C}^{+})$$

Example (1.5 in the book Yoneda) ← will not cover this in my talk

Note that both

$$\begin{tikzcd} \mathrm{Fun}([1],C) \arrow[d, "ev_0"'] \\ C \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C \times C \arrow[d, "proj_1"] \\ C \end{tikzcd}$$

are cartesian fibrations.

An ev_0-cartesian morphism in $\mathrm{Fun}([1],C)$ is a morphism $\alpha : f \to f'$ s.t. $ev_1(\alpha) = \alpha_1$ is an isomorphism.

$$\Updownarrow$$

$$\begin{tikzcd} c_0 \arrow[r, "\alpha_0"] \arrow[d, "f"'] & c_0' \arrow[d, "f'"] \\ c_1 \arrow[r, "\alpha_1"'] \arrow[ur, phantom, "\simeq"] & c_1' \end{tikzcd}$$

Analogously: $proj_1$-cartesian morphism is a morphism $(\alpha_0, \alpha_1) : (c_0, c_1) \to (c_0', c_1')$ such that $proj_2(\alpha_0, \alpha_1) = \alpha_1$ is an iso.

In particular, the commutative triangle

$$\begin{tikzcd} \mathrm{Fun}([1],C) \arrow[rr, "(ev_0, ev_1)"] \arrow[dr, "ev_0"'] & \simeq & C \times C \arrow[dl, "proj_1"] \\ & C & \end{tikzcd}$$

is a morphism in (Cart/C) seriel $\simeq \mathrm{Fun}(C^{op}, 1\mathrm{Cat})$

(ev_0, ev_1) clearly sends ev_0-cartesian morphisms to $proj_1$-cartesian morphisms

Straightening this morphisms yields a natural transformation

$$\circledast \quad (c \mapsto C_{c/}) \Longrightarrow \underline{C}$$

constant diagram functor

$$C^{op} \to 1Cat$$
$$x \mapsto C$$

components are

$$C_{c/} \xrightarrow{ev_1} C$$

by def of straightening

will not speak about this

Note that:

$$\mathrm{Fun}(C^{op}, 1Cat)_{/\underline{C}} \xrightarrow{\simeq} \mathrm{Fun}(C^{op}, 1Cat_{/C})$$

and hence

$\circledast$ gives us a functor

$$(**) \quad C^{op} \to 1Cat_{/C}$$
$$c \mapsto (\underbrace{C_{c/} \to C}_{\substack{0\text{-CoCartesian fib} \\ \text{w/ fiber } \mathrm{Map}_C(c,x) \in Spc \\ \text{at } x}})$$

and therefore $(**)$ factors as

$$\begin{tikzcd} C^{op} \arrow[r] \arrow[dr, dashed, "\simeq"'] & 1Cat_{/C} \\ & 0\text{-}CoCart_{/C} \arrow[u, "incl"'] \end{tikzcd}$$

$$0\text{-}CoCart_{/C} \simeq \mathrm{Fun}(C, Spc)$$

So by currying we get the Hom functor

$$C^{op} \times C \xrightarrow{\mathrm{Map}_C} Spc.$$

$\underbrace{1.6}_{\text{in the book}}$ gives an enhanced Straightening/Unstraightening equivalence

$$(\circledast)\qquad \begin{tikzcd} \mathrm{CoCart} \arrow[d, hook, "\text{full}"] \arrow[dd, bend right=60, "ev_1"'] \\ \mathrm{Fun}([1], 1\mathrm{Cat}) \arrow[d, "ev_1"] \\ 1\mathrm{Cat} \end{tikzcd}$$

is a Cartesian fibration.

Analogously

$$\begin{tikzcd} \mathrm{Cart} \arrow[d] \\ 1\mathrm{Cat} \arrow[d, "\wr"] \\ B \mapsto \mathrm{Fun}(B^{op}, 1\mathrm{Cat}) \simeq (\mathrm{Cart}_{/B})^{\text{strict}} \end{tikzcd}$$

$\circledast$ classifies

$$1\mathrm{Cat}^{op} \longrightarrow \widehat{1\mathrm{Cat}}$$
$$B \longmapsto \mathrm{Fun}(B, 1\mathrm{Cat}) \simeq (\mathrm{CoCart}_{/B})^{\text{strict}}$$

makes precise that Straightening/Unstraightening is contravariantly natural in B...

1.7 Adjoint functors

$F : C_0 \to C_1$ functor

$\Longleftrightarrow [1] \xrightarrow{F} 1Cat$

unstraighten

$\Longleftrightarrow \tilde{C} \xrightarrow{\tilde{F}} [1]$ CoCartesian fibration

def $F : C_0 \to C_1$ admits a right adjoint

if $\tilde{C} \xrightarrow{\tilde{F}} [1]$ is also a Cartesian

fibration (i.e if is biCartesian).

Assume $F: C_0 \to C_1$ admits a RA. Then we may apply cartesian unstraightening to

$$\tilde{C} \xrightarrow{\tilde{F}} [1]$$ to obtain

$$\begin{array}{ccc} [1]^{op} & \longrightarrow & 1Cat \\ 0 & \longmapsto & C_0 \\ \uparrow & \longmapsto & \uparrow F^R \\ 1 & \longmapsto & C_1 \end{array}$$

hence we get a functor $C_1 \xrightarrow{F^R} C_0$.

Exercise: Show that for all $x \in C_0$, $y \in C_1$ one has

$$\mathrm{Map}_{C_1}(Fx, y) \simeq \mathrm{Map}_{C_0}(x, F^R y)$$

1.7.3 Partial Adjoints

$\tilde{C} \xrightarrow{\tilde{F}} [1]$ as before without assuming $\tilde{F}$ is a cartesian fibration.

Let $C_1' \subset C_1$ full subcat. on those $e_1 \in C_1$ such that there exists an $\tilde{F}$-cartesian morphism $c_0 \longrightarrow c_1$ in $\tilde{C}$ covering $0 \to 1$ in $[1]$.

Let $\tilde{C}' \underset{\text{full}}{\subset} \tilde{C}$ full subcat so that

$$\tilde{C}_0' = C_0$$
$$\tilde{C}_1' = C_1'$$

Then :

$\tilde{\mathcal{C}}' \longrightarrow [1]$ is a Cartesian fibration

$\Longrightarrow$ partially defined right adjoint

$$\mathcal{C}'_1 \xrightarrow{F'^R} \mathcal{C}_0$$

$$\mathrm{Map}_{\mathcal{C}_1}(Fc_0, c_1) \simeq \mathrm{Map}_{\mathcal{C}_0}(c_0, F'^R c_1)$$

for all $c_0 \in \mathcal{C}_0$
$c_1 \in \mathcal{C}'_1$

2. Basic Operations on ∞-Cats

Q1 How to produce new ∞-Cats from old ones? (e.g. derived Cats)

Q2 How to construct new functors from old ones?

A1 (co)limits e.g. localizations realised as a pushout
$\to$ derived ∞-Cat

A2 Kan extensions e.g. construction of $\mathcal{Q}\mathrm{Coh}(-)$

2.1.1 $F : D \to C$

For any category E we have

$$\mathrm{Fun}(C, E) \xrightarrow{F^*} \mathrm{Fun}(D, E)$$

precomposition functor

2.1.2

def partially defined left (resp. right) adjoint of F^* is called left (resp. right) Kan extension along F and denoted by

LKE_F (resp. RKE_F)

Example We will see many examples of Kan extensions in future sessions;

Easy examples are given by

(co) limits :

If $C = *$, $F = (D \xrightarrow{!} *)$

Then left and right Kan extension give

$$\operatorname*{colim}_{D} : \mathrm{Fun}(D, E) \longrightarrow E$$

$$\lim_{D} : \mathrm{Fun}(D, E) \longrightarrow E$$

(partially defined)

2.1.3 Formulas for Kan extensions

Let

$$\begin{tikzcd} D \arrow[r, "\Phi"] \arrow[d, "F"'] & E \\ C & \end{tikzcd}$$

Consider for $c \in C$

$$\begin{tikzcd} F\downarrow c \simeq D \times_C C_{/c} \arrow[r] \arrow[d, "\pi_D"'] \arrow[dr, phantom, "\lrcorner", very near start] & C_{/c} \arrow[d] \\ D \arrow[r, "F"'] & C \end{tikzcd}$$

objects : $Fd \xrightarrow{\alpha} c$

morphisms : $\begin{array}{c} d \\ \lambda \downarrow \\ \tilde{d} \end{array}$ $\begin{tikzcd} Fd \arrow[rr, dashed, "F\lambda"] \arrow[dr, "\alpha"'] & \simeq & F\tilde{d} \arrow[dl, "\tilde{\alpha}"] \\ & c & \end{tikzcd}$

and suppose that for every $c \in C$

notation$\left\{ \operatorname{colim}_{D \times_C C_{/c}} \Phi \right. := \operatorname{colim}_{D \times_C C_{/c}} (\Phi \circ \pi_D)$ exists.

Then : $\mathrm{LKE}_F \Phi : C \to E$ exists and on objects is given by

$$c \longmapsto \operatorname*{colim}_{D \times_C C_{/c}} \Phi \simeq \mathrm{LKE}_F \Phi(c)$$

Similarly :

$$\mathrm{RKE}_F \Phi(c) \simeq \lim_{D \times_C C_{c/}} \Phi$$

Exercise :

$$\operatorname*{colim}_D \Phi \simeq \operatorname*{colim}_C \mathrm{LKE}_F(\Phi)$$

$$\lim_D \Phi \simeq \lim_C \mathrm{RKE}(\Phi)$$

2.1.5 There is an adjunction

$$\begin{tikzcd} \mathrm{Spc} \arrow[r, hook] & 1\mathrm{Cat} \arrow[l, bend right, "|-|"'] \arrow[l, bend left, "(-)^{\simeq}"] \end{tikzcd}$$

$|-|$: freely invert 1-morphisms

$(-)^{\simeq}$: maximal subspace

Note that

$$|C| \simeq \operatorname{colim}_C \underline{*}$$

constant diagram functor
$C \to \mathrm{Spc}$
$c \mapsto *$

For any $S \in \mathrm{Spc}$ we have
$|S| \simeq S$.

Indeed, this may be proven by straightening-unstraightening:

will not come up in lecture

$$\begin{aligned} &\mathrm{Map}_{\mathrm{Spc}}(\operatorname{colim}_C \underline{*}, S) \\ &\simeq \lim_C \mathrm{Map}_{\mathrm{Spc}}(*, S) \\ &\simeq \lim_C \underline{S} \simeq \Gamma(C \times S \to C) \\ &\simeq \mathrm{Fun}(C, S) \\ &\simeq \mathrm{Map}_{\mathrm{Spc}}(C, S) \end{aligned}$$

as wanted

def $C : 1Cat$ is called contractible if

$$|C| \simeq *$$

2.1.6 If $\Phi : C \to \mathrm{Spc}$, then

$$\operatorname*{colim}_C \Phi \simeq |\tilde{C}_\Phi| \quad \text{where}$$

$\tilde{C}_\Phi \to C$ is the coCartesian fib. assoc. to Φ.

Lemma

$\mathcal{C} \in 1\mathrm{Cat}$, $\quad \mathcal{P}\mathcal{C} := \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Spc})$

Then restriction and left Kan extension along the Yoneda embedding

$$\mathcal{C} \overset{\text{よ}}{\hookrightarrow} \mathcal{P}\mathcal{C}$$

induce for all cocomplete $\mathcal{D}$ an equivalence

$$\mathrm{Fun}^{\mathrm{colim}}(\mathcal{P}\mathcal{C}, \mathcal{D}) \underset{\mathrm{LKE}_{\text{よ}}}{\overset{\text{よ}^*}{\underset{\longleftarrow}{\overset{\sim}{\longrightarrow}}}} \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

exhibiting $\mathcal{P}\mathcal{C}$ as the **free cocompletion** of $\mathcal{C}$.

In particular: $\quad \mathrm{Fun}^{\mathrm{colim}}(\mathrm{Spc}, \mathcal{D}) \simeq \mathcal{D}$.

§3 Cofinality

2.2.1

def

$F: D \to C$ is called cofinal if for any $c : C$, the category

$c \downarrow F \simeq D \times_C C_{c/}$ is contractible i.e.

$$|D \times_C C_{c/}| \simeq *$$

(and coinitial if $F^{op} : D^{op} \to C^{op}$ is cofinal)

i.e $|F \downarrow c| \simeq *$

2.2.2

Lemma TFAE:

(i) $F : D \to C$ is cofinal.

(ii) For any $\Phi : C \to E$, the natural map

$$\operatorname{colim}_D (\Phi \circ F) \xrightarrow[\sim]{} \operatorname{colim}_C \Phi$$

is an isomorphism.

(iii) For any $\Phi : C^{op} \to E$, the natural map

$$\lim_{C^{op}} \Phi \xrightarrow{\sim} \lim_{D^{op}} (\Phi \circ F^{op})$$

is an iso.

In the book there is also: (ii'), (ii''), (iii'), (iv)

2.2.3

Example

- Let C be a category w/ a terminal object $t \in C$. Then the functor

$$* \xrightarrow{t} C$$ is cofinal i.e

for $\Phi : C \to E$ we have

$$\operatorname{colim}_{C} \Phi \simeq \operatorname{colim}_{*} \Phi(t) \simeq \Phi(t)$$

- More generally: Any functor

$$F : D \to C$$ that admits a left adjoint (i.e F is a right adjoint) is cofinal.

Indeed: $D \times_{C} C_{c/} \simeq c \downarrow F$ has an initial object for all c and hence $|D \times_{C} C_{c/}| \simeq *$.

Example / Lemma

$D \xrightarrow{F} C$ coCartesian fibration

Then the canonical functor

$$d \longmapsto (c, \mathrm{id}_c)$$

$$\begin{tikzcd}
D_c \arrow[dr, dashed, "\tilde{F}"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& D \times_C C_{/c} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C_{/c} \arrow[d] \\
& D \arrow[r, "\Phi"'] & C
\end{tikzcd}$$

(i.e $\tilde{F} : D_c \to D \times_C C_{/c}$

$d \longmapsto (d, \mathrm{id}_c)$)

is cofinal.

See Kerodon Prop 7.3.4.1

Hence we get:

$$LKE_F \Phi(c) \simeq \operatorname*{colim}_{D \times C_{/c}} \Phi \simeq \operatorname*{colim}_{D_c} \Phi$$

Similarily, if $D \xrightarrow{F} C$ is a Cartesian fib.

$$RKE_F \Phi(c) \simeq \lim_{D_c} \Phi$$

2.3 The book also says something on contractible functors ...
we will skip this for now